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The quantum general linear supergroup and braid statistics

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Abstract. In this paper we generalize to braid statistics some of Manin's work on the quantum deformation of the general linear supergroup. The key ingredient in this construction is the introduction of a non-standard transposition map Ψ ($\Psi^2 \neq 1$) which is defined in terms of a generalized permutation operator P_μ ($P_\mu^2 \neq 1$). The dual space (enveloping algebra) with P_μ-statistics is defined. Consideration of coactions on quantum spaces clarifies the resulting structure.

1. Introduction

The Perk–Schultz model (generalized six-vertex) and its associated quantum spin chains are of current interest to the condensed matter community [1, 2]. The statistical weights of this model and the corresponding solutions $\check{R}$ of the braid relation are given in equations (2.5b) and (2.5a), respectively. Using the quantum inverse scattering method (QISM), de Vega and Lopes [3, 4] obtained an exact solution. There have been numerous studies of the underlying mathematical structure of this model, i.e. of the quantum groups and enveloping algebras associated with the matrices (2.5a). Two types of structure have emerged from these studies. On the one hand the R-matrices (2.5a) are by now well known to be related to the deformation of the general linear supergroups $GL(M|N)$ and their enveloping algebras [5–13]. The use of the graded version of the Faddeev–Reshetikhin–Takhtajan (FRT) formalism will lead to such structures. On the other hand, quantum algebras can be obtained by taking the ultrarelativistic limit [14] of the Yang–Baxter algebras used in the QISM. Based on the work by de Vega and Lopez [3, 4] one would expect to obtain non-graded algebras from this process. In fact their work suggests the use of the non-graded version of the FRT formalism [15] to construct the quantum groups and algebras associated with the R-matrices (2.5a). This, in fact, has been done by several authors [9, 11, 16–19] and algebraic structures that resemble the super ones, but which have no classical limit, have been obtained. The claim that two distinct algebraic structures are associated with the same R-matrix is somewhat puzzling and raises the question of their relationship. In order to understand the key difference between these two types of algebra, let us recall that in describing the coproduct Δ and antipode S of a Hopf algebra H, one must specify the transposition map $\Psi : a_1 \otimes a_2 \to a_2 \otimes a_1$, where $a_1, a_2 \in H$. In the case of the coproduct, Ψ appears in the definition of the multiplication rule in $H \otimes H$ (see (2.3)) and for the antipode it enters through the relation (3.27). While the superalgebras are characterized by a graded transposition map (superstatistics) $\Psi_s(a_1 \otimes a_2) = (-1)^{p(a_1)p(a_2)}(a_2 \otimes a_1)$, where $p(\alpha)$ denotes the parity of element α, the algebras obtained from the non-graded version of the FRT formalism are described in terms of the map $\Psi(a_1 \otimes a_2) = (a_2 \otimes a_1)$ (Bose statistics).

In this paper we show, through a generalization of the FRT formalism, that the quantum algebraic structures associated with the family of R-matrices given in (2.5a) may be defined in the more general framework of Majid's braided tensor categories [20, 21]. These new structures may be interpreted as the result of two types of deformation: the usual q-deformation associated with the R-matrices given in (2.5a) combined with what may be viewed as a μ-deformation of the statistics (by deforming Ψ) associated with another R-matrix, namely the generalized permutation operator $[P_\mu]^{cd}_{ab} = \delta^d_a\delta^c_b\mu_{cd}$, with the μ_{cd}'s $(c, d = 1, n)$ being arbitrary parameters. Since Ψ is defined in terms of a solution (P_μ) of the braid relation, we follow Majid and say that such structures obey braid statistics. In the light of these results, the structures obtained from the graded and non-graded versions of the FRT formalism discussed earlier are seen to be only two points on a continuum of possibilities. Let us add that, to our knowledge, the general concept of braided tensor categories was first introduced by Joyal and Street [22]. The key ingredient in our generalization is the introduction of a non-standard transposition operator $\Psi : V\otimes W \to W\otimes V$, where V and W are two vector spaces, which includes as a special case the transpositions associated with the Bose and superstatistics. It is non-standard in the sense that $\Psi^2 \neq 1$ in general. In tackling this problem we adopt Manin's [23] viewpoint and consider quantum groups as symmetries of quantum spaces. His approach may be viewed as a more fundamental construction than the FRT formalism [15], enforcing tighter constraints on the allowed structures and also providing more direct checks of consistency. In particular, the tensorial structure underlying the entire formalism can be more explicitly examined and generalized.

Although of more general utility, our results are presented as a generalization to the braid statistics of Manin's one-parameter deformation of the general linear supergroup [23]. A generalization of the FRT formalism to braid statistics naturally follows. We use his notation in describing the Z_2 grading of various objects. The Z_2 degree of an element b will be denoted $\hat{b}$; a format is an arbitrary sequence $(a_1, a_2, \ldots, a_n)$ with $a_i \in Z_2$. Every algebra (whose elements are characterized by one (c_i) or two indices (α^j_i)) will be associated with a given format by defining the grading $\hat{\alpha}^j_i$ of its elements α^j_i $(i, j = 1, \ldots, n)$ as $\hat{\alpha}^j_i = a_i + a_j$ $(\hat{c}_i = a_i$ or $1 - a_i)$. Putting $\hat{\imath} = a_i$ we then get $\hat{\alpha}^j_i = \hat{\imath} + \hat{\jmath}$ $(\hat{c}_i = \hat{\imath}$ or $1 - \hat{\imath})$. Throughout this paper $A(n)$ denotes an algebra over a fixed field k associated with the format $(a_1, a_2, \ldots, a_n)$.

Finally, we conclude this introduction by stressing the fact that, since the FRT formalism is based on the main relations of the QISM, our generalization of this formalism to braid statistics was motivated by its possible implications at the level of certain integrable systems. The results of this paper strongly suggest the existence of a braided version of the QISM. In such a formulation the condition of ultralocality [24] would no longer be imposed. Based on the results described below, we proposed [25] for spectral dependent R-matrices of the charge conserved type (see definition below (2.5a)), such as those of the Perk–Schultz model, a braided version of the main relations of the QISM and proved the existence of a family of commuting transfer operators. Further work is required and, at present, it is not clear what role such additional structure (braiding) would have in these models.

The paper is organized as follows. In section 2 we recall, for ease of reference, some known results on the one-parameter deformation of the general linear supergroup and its enveloping algebra. The main results of this paper are stated in theorem 1 of section 3 and theorem 3 of section 4 in which we prove the existence of bialgebras $M_{q,\mu}(n)$ and $U_{q,\mu}(n)$ (in the dual space) that are characterized by special braid statistics (P_μ-statistics). In section 5 we consider in some detail a two-dimensional example. Some of the results presented in this paper were announced in [25–27].

2. Quantum general linear supergroup and its dual space

Let us denote by $E_q(n)$ the Z_2-graded- or super-bialgebra generated by the elements z_i^j $(i, j = 1, \ldots, n)$ and unit element **1** which satisfy the following relations [23]

$$
\begin{aligned}
&(z_i^k)^2 = 0 && \text{for } \hat{\imath} + \hat{k} = \text{odd} \\
&z_i^k z_i^\ell - (-1)^{\hat{k}\hat{\ell}} q^{-1} z_i^\ell z_i^k = 0 && \text{for } \hat{\imath} = \text{even},\ k < \ell \\
&z_i^k z_i^\ell - (-1)^{(\hat{k}+1)(\hat{\ell}+1)} q z_i^\ell z_i^k = 0 && \text{for } \hat{\imath} = \text{odd},\ k < \ell \\
&z_i^k z_j^k - (-1)^{\hat{\imath}\hat{\jmath}} q^{-1} z_j^k z_i^k = 0 && \text{for } \hat{k} = \text{even},\ i < j \\
&z_i^k z_j^k - (-1)^{(\hat{\imath}+1)(\hat{\jmath}+1)} q\ z_j^k z_i^k = 0 && \text{for } \hat{k} = \text{odd},\ i < j \\
&z_i^k z_j^\ell - (-1)^{(\hat{\jmath}+\hat{\ell})(\hat{\imath}+\hat{k})} z_j^\ell z_i^k = (-1)^{\hat{\imath}\hat{\jmath}+\hat{k}\hat{\imath}+\hat{k}\hat{\jmath}} (q^{-1} - q) z_j^k z_i^\ell \\
& && \text{for } i < j, k < \ell \\
&z_i^\ell z_j^k = (-1)^{(\hat{\imath}+\hat{\ell})(\hat{k}+\hat{\jmath})}\ z_j^k z_i^\ell && \text{for } i < j, k < \ell.
\end{aligned}
\tag{2.1}
$$

The coproduct $\Delta : E_q(n) \to E_q(n) \otimes E_q(n)$ and counit $\epsilon : E_q(n) \to k$ are homomorphisms given by

$$\Delta(z_i^j) = \sum_{k=1}^{n} z_i^k \otimes z_k^j \qquad \epsilon(z_i^j) = \delta_i^j.$$

The defining axioms of a superbialgebra are essentially the same as those of a non-graded bialgebra. The main ingredient that distinguishes them is the graded transposition map Ψ_s, which is defined as

$$\Psi_s(a \otimes b) = (-1)^{\hat{a}\hat{b}} (b \otimes a) \qquad a, b \in E_q(n).$$

For $a = z_i^j$ and $b = z_k^\ell$ we thus have

$$\Psi_s(z_i^j \otimes z_k^\ell) = (-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{\ell})} (z_k^\ell \otimes z_i^j).$$

The action of a family of generalized transposition maps [20, 21] involving higher tensor products is governed by

$$
\begin{aligned}
\Psi((a \otimes b) \otimes c) &= (\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(a \otimes b \otimes c) \\
\Psi(a \otimes (b \otimes c)) &= (\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id})(a \otimes b \otimes c)
\end{aligned}
\tag{2.2a}
$$

(where a, b and c lie in arbitrary spaces, and the use of the same symbol Ψ for maps on different spaces should cause no confusion), with the transposition on composites then given by

$$
\begin{aligned}
\Psi(ab \otimes c) &= (\mathrm{id} \otimes m)\Psi((a \otimes b) \otimes c) \\
\Psi(a \otimes bc) &= (m \otimes \mathrm{id})\Psi(a \otimes (b \otimes c))
\end{aligned}
\tag{2.2b}
$$

where m is the multiplication map in the appropriate space. With some abuse of notation the product rule in a tensor product of spaces $\mathcal{A}\otimes\mathcal{B}$ is defined in terms of $\Psi : \mathcal{B}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$ as

$$(a\otimes b)(c\otimes d) = a\Psi(b\otimes c)d \qquad a, c \in \mathcal{A},\ b, d \in \mathcal{B} \tag{2.3}$$

where we transpose b and c in the generalized sense defined by Ψ and then multiply the adjacent factors in $\mathcal{A}$ and $\mathcal{B}$ separately. This will be understood without further comment for all the transposition maps considered in this paper. Thus in $E_q(n)\otimes E_q(n)$, for example, we have that, for generators z_i^j

$$(z_i^j \otimes z_k^\ell)(z_m^p \otimes z_r^s) = (-1)^{(\hat{k}+\hat{\ell})(\hat{m}+\hat{p})}(z_i^j z_m^p \otimes z_k^\ell z_r^s).$$

Manin has also formally added a Hopf structure to $E_q(n)$ by defining an antipode S_s which is required to be a graded antihomomorphism

$$S_s(ab) = (-1)^{\hat{a}\hat{b}} S_s(b) S_s(a).$$

The relations (2.1) can be written [9] as

$$\sum_{i,j=1}^{n} \check{R}_{ab}^{ij} z_i^k z_j^\ell (-1)^{\hat{k}(\hat{j}+\hat{\ell})} = \sum_{i,j=1}^{n} z_a^i z_b^j \check{R}_{ij}^{k\ell}(-1)^{\hat{i}(\hat{b}+\hat{j})}$$

or in a more compact way as the matrix product

$$\check{R} Z_1 P_s Z_1 P_s = Z_1 P_s Z_1 P_s \check{R} \tag{2.4}$$

where P_s is the graded permutation operator with $[P_s]_{ab}^{cd} = \delta_a^d \delta_b^c (-1)^{\hat{c}\hat{d}}$, Z is the $n\times n$ matrix $(Z)_i^j = z_i^j$ and $Z_1 = Z\otimes I$ (where the tensor product is not in $E_q(n)\otimes E_q(n)$, but in the space of matrices, i.e. $[Z_1]_{ab}^{cd} = Z_a^c \delta_b^d$). $\check{R}$ is the following matrix solution of the braid relation

$$\check{R} = (1-q^2)\sum_{1\leqslant i<j\leqslant n} e_i^i \otimes e_j^j + \sum_{i=1}^{n}(-1)^{\hat{i}} q^{2\hat{i}} e_i^i \otimes e_i^i + q\sum_{i\neq j}(-1)^{\hat{i}\hat{j}} e_i^j \otimes e_j^i \tag{2.5a}$$

where e_i^j is a matrix unit: $(e_i^j)_k^\ell = \delta_{ik}\delta_{j\ell}$. Note that the only non-zero matrix elements of $\check{R}$ are of the type $\check{R}_{ii}^{ii}$ (all i), $\check{R}_{ij}^{ij}$ $(i<j)$ and $\check{R}_{ij}^{ji}$ (all i, j); we shall refer to this property as charge conservation. Note [9] that one can transform (Baxterize) this matrix into the matrix solution of the spectral braid relation

$$\begin{aligned}\check{R}(\theta;\nu) = {} & \sum_{\substack{a\neq b\\ a,b=1}}^{n} e^{i\theta\,\mathrm{sgn}(a-b)} e_a^a \otimes e_b^b + \sum_{a=1}^{n}\sin(\nu + (-1)^{\hat{a}}\theta)\sin(\nu)^{-1} e_a^a\otimes e_a^a \\ & + \sum_{\substack{a\neq b\\ a,b=1}}^{n}(-1)^{\hat{a}\hat{b}}\sin(\theta)\sin(\nu)^{-1} e_b^a \otimes e_a^b \end{aligned} \tag{2.5b}$$

whose elements are the statistical weights of the generalized six-vertex model (in the trigonometric regime). Here $q = e^{i\nu}$ and θ is the spectral parameter.

The fundamental (+) and conjugate fundamental (−) representations $\rho^{(\pm)}$ of $E_q(n)$ are defined by

$$\rho^{(\pm)}(z_i^j)_\alpha^\beta = (\check{R}^{\pm 1})_{\alpha i}^{j\beta}(-1)^{\hat{\jmath}\hat{\beta}}.$$

As was shown in [23], $E_q(n)$ coacts on a pair $\mathcal{A}_q, \mathcal{A}_q^*$ of quantum spaces. $\mathcal{A}_q$ is an algebra generated by n coordinates x_i $(i = 1, \ldots, n)$ with parity assignment $\hat{x}_i = \hat{\imath}$ and quadratic relations ('commutation relations')

$$(x_i)^2 = 0 \qquad \text{for } \hat{\imath} = 1 \tag{2.6}$$

$$x_i x_j - (-1)^{\hat{\imath}\hat{\jmath}} q^{-1} x_j x_i = 0 \qquad \text{for } i < j. \tag{2.7}$$

$\mathcal{A}_q^*$ is generated by n coordinates ξ_i $(i = 1, \ldots, n)$ with parity assignment $\hat{\xi}_i = 1 - \hat{\imath}$ and commutation relations

$$(\xi_i)^2 = 0 \qquad \text{for } \hat{\imath} = 0$$

$$\xi_i \xi_j - q(-1)^{(\hat{\imath}+1)(\hat{\jmath}+1)} \xi_j \xi_i = 0 \qquad \text{for } i < j.$$

By requiring that the maps

$$\delta(x_i) = \sum_{j=1}^{n} z_i^j \otimes x_j \qquad \delta^*(\xi_i) = \sum_{j=1}^{n} z_i^j \otimes \xi_j$$

be homomorphisms of $\mathcal{A}_q$ and $\mathcal{A}_q^*$, respectively, and use of the following transposition maps

$$\Psi(c_i \otimes z_k^\ell) = (-1)^{\hat{c}_i + \hat{z}_k^\ell}(z_k^\ell \otimes c_i)$$

where $c_i = x_i$ or ξ_i, one obtains, for $q^2 \neq -1$, the relations (2.1). Note that one could have (this will be convenient in the next section) considered the coaction on a set $(Q_1(n), Q_2(n))$ where $Q_1(n) = \mathcal{A}_q$ and $Q_2(n)$ is the quadratic algebra with n coordinates y_i $(i = 1, \ldots, n)$ and parity assignment $\hat{y}_i = \hat{\imath}$ satisfying the relations

$$(y_i)^2 = 0 \qquad \text{for } \hat{\imath} = 0 \tag{2.8}$$

$$y_i y_j + (-1)^{\hat{\imath}\hat{\jmath}} q y_j y_i = 0 \qquad \text{for } i < j \tag{2.9}$$

which are now not of 'supersymmetric' form. In both cases one obtains the set of relations (2.1), and it turns out that both $Q_2(n)$ and $\mathcal{A}_q^*$ may be viewed as being dual to $\mathcal{A}_q$, albeit with different bilinear pairings. The choice $(Q_1(n), Q_2(n))$ is, however, the natural one, being representative of the more general situation (i.e. for more general $\check{R}$, for which one may need several spaces $Q_i(n)$ not related by duality [28]).

We now turn to the dual space [9, 15]. Let us denote by $U_q(n)$ that subalgebra of the dual to $E_q(n)$ which is generated by the unit element $\mathbf{1}'$ and the generators $L^{(\pm)}{}_i^j$ $(i, j = 1, \ldots, n)$ which are defined by duality relations which may be written in terms of tensor products in matrix space as

$$\langle \mathbf{1}' | Z_1 Z_2 \cdots Z_k \rangle = I^{\otimes k} \qquad \langle \mathbf{1}' | \mathbf{1} \rangle = 1$$

$$\langle L^{(\pm)} | Z_1 Z_2 \cdots Z_k \rangle = R_1^{(\pm)} R_2^{(\pm)} \cdots R_k^{(\pm)} \qquad \langle L^{(\pm)} | \mathbf{1} \rangle = I$$

where $L^{(\pm)} = (L^{(\pm)}{}_i^j)$ and $I = (\delta_i^j)$ are $n \times n$ matrices and $Z_i = I \otimes I \cdots \otimes Z \cdots \otimes I$ (Z in the ith position of k factors). $R_i^{(\pm)}$ acts on factors number 0 (corresponding to the $L^{(\pm)}$-space) and i, and coincides there with $R^{(\pm)} = \check{R}^{\pm 1} P_s$. The generators satisfy the quadratic relations

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} L^{(\epsilon)}{}_j^k L^{(\epsilon')}{}_i^\ell (-1)^{\hat{k}(\hat{\imath}+\hat{\ell})} = \sum_{i,j=1}^{n} L^{(\epsilon')}{}_b^i L^{(\epsilon)}{}_a^j \check{R}^{\ell k}_{ji} (-1)^{\hat{\imath}(\hat{a}+\hat{\jmath})}$$

where $(\epsilon, \epsilon') = (\pm, \pm)$ or $(+, -)$. The coproduct and counit are given by

$$\Delta(L^{(\pm)}{}_i^j) = \sum_{k=1}^{n} L^{(\pm)}{}_i^k \otimes L^{(\pm)}{}_k^j \qquad \epsilon(L^{(\pm)}{}_i^j) = \delta_i^j .$$

The grading of $L^{(\pm)}{}_i^j$ is taken to be $\widehat{L^{(\pm)}{}_i^j} = \hat{\imath} + \hat{\jmath}$ and transposition is defined as in $E_q(n) \otimes E_q(n)$ by the map

$$\Psi(L^{(\epsilon)}{}_i^j \otimes L^{(\epsilon')}{}_k^\ell) = (-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{\ell})} (L^{(\epsilon')}{}_k^\ell \otimes L^{(\epsilon)}{}_i^j).$$

A representation ρ is yielded by

$$\rho(L^{(+)}{}_i^j)_\alpha^\beta = \check{R}^{\beta j}_{i\alpha} (-1)^{\hat{\beta}\hat{\jmath}} \qquad \rho(L^{(-)}{}_i^j)_\alpha^\beta = (\check{R}^{-1})^{\beta j}_{i\alpha} (-1)^{\hat{\beta}\hat{\jmath}}.$$

In the next section, we generalize the considerations of this section and prove the existence of a bialgebra $M_{q,\mu}(n)$ which is characterized by statistics that are more general than the supersymmetric or Z_2-graded ones of Manin's $E_q(n)$, but which includes $E_q(n)$ as a special case.

3. Quantum symmetries and P_μ-statistics

Let us denote by $M_q(n)$ the associative algebra whose n^2 generators t_i^j $(i, j = 1, \ldots, n)$ satisfy the commutation rules

$$(t_i^k)^2 = 0 \qquad \text{for } \hat{\imath} + \hat{k} = \text{odd} \tag{3.1}$$

$$t_i^k t_i^\ell (k|i\ell) - (-1)^{\hat{k}\hat{\ell}} q^{-1} t_i^\ell t_i^k (\ell|ik) = 0 \qquad \text{for } \hat{\imath} = \text{even},\ k < \ell \tag{3.2}$$

$$t_i^k t_i^\ell (k|i\ell) + q(-1)^{\hat{k}\hat{\ell}} t_i^\ell t_i^k (\ell|ik) = 0 \qquad \text{for } \hat{\imath} = \text{odd},\ k < \ell \tag{3.3}$$

$$t_i^k t_j^k (k|jk) - (-1)^{\hat{\imath}\hat{\jmath}} q^{-1} t_j^k t_i^k (k|ik) = 0 \qquad \text{for } \hat{k} = \text{even},\ i < j \tag{3.4}$$

$$t_i^k t_j^k (k|jk) + (-1)^{\hat{\imath}\hat{\jmath}} q\ t_j^k t_i^k (k|ik) = 0 \qquad \text{for } \hat{k} = \text{odd},\ i < j \tag{3.5}$$

$$t_i^k t_j^\ell (k|j\ell) - (-1)^{\hat{\imath}\hat{\jmath}+\hat{k}\hat{\ell}}\ t_j^\ell t_i^k (\ell|ik) = (-1)^{\hat{\imath}\hat{\jmath}} (q^{-1} - q) t_j^k t_i^\ell (k|i\ell) \qquad \text{for } i < j, k < \ell \tag{3.6}$$

$$t_i^\ell t_j^k (\ell|jk) = (-1)^{\hat{\imath}\hat{\jmath}+\hat{\ell}\hat{k}} (k|i\ell)\ t_j^k t_i^\ell \qquad \text{for } i < j, k < \ell \tag{3.7}$$

where the coefficients $(a|bc)$ are for now arbitrary functions of the indices. These relations can be written more succinctly in terms of the R-matrix (2.5a) as

$$\sum_{i,j=1}^{n} \check{R}_{ab}^{ij} t_i^k t_j^\ell (k|j\ell) = \sum_{i,j=1}^{n} t_a^i t_b^j \check{R}_{ij}^{k\ell} (i|bj). \tag{3.8}$$

To understand the origin of these equations, let us now define a transposition map Ψ: $Q_i(n) \otimes M_q(n) \to M_q(n) \otimes Q_i(n)$ through

$$\Psi(c_i \otimes t_k^\ell) = (i|k\ell)(t_k^\ell \otimes c_i) \tag{3.9}$$

with the quantum spaces $Q_i(n)$ defined in section 2 and c_i being x_i or y_i as appropriate. Accordingly, the $(i|jk)$ will be referred to as the three-index transposition coefficients. Let us recall that there has been so far, in the literature, two types of algebra associated with the family of R-matrices given in (2.5a). One is characterized by superstatistics [5–13], while the other has Bose statistics [9, 11, 16–19]. This raises the question of the existence of a more general structure which would include these two types as special cases ($(i|jk) = 1$ or $(i|jk) = (-1)^{\hat{\imath}(\hat{\jmath}+\hat{k})}$). It is the search for such a structure which motivated our introduction of the three-index coefficients.

Proposition 1. For $q^2 \neq -1$, the maps $\delta_i : Q_i(n) \to M_q(n) \otimes Q_i(n)$

$$\delta_1(x_i) = \sum_{j=1}^{n} t_i^j \otimes x_j \qquad \delta_2(y_i) = \sum_{j=1}^{n} t_i^j \otimes y_j \tag{3.10}$$

define a left-coaction of the algebra $M_q(n)$ on the algebras $Q_1(n)$ and $Q_2(n)$.

Proof. What needs to be shown is that δ_1 and δ_2 are homomorphisms of $Q_1(n)$ and $Q_2(n)$, respectively. Applying these maps on the left-hand sides of equations (2.6) and (2.7) ((2.8) and (2.9)) and using the multiplication rule following from (3.9), one must calculate the coefficients of the monomials $1 \otimes x_k x_\ell$ $(1 \otimes y_k y_\ell)$ $k \leqslant \ell$ which are linearly independent over M_q in $M_q \otimes Q_1(n)$ $(M_q \otimes Q_2(n))$. Application of δ_1 on (2.6) gives the following: the coefficient of $1 \otimes (x_k)^2$ gives relation (3.1) for $\hat{\imath} = 1$ and $\hat{k} = 0$, while relation (3.3) is obtained from the coefficient of $1 \otimes x_k x_\ell$ and for $k < \ell$. Application of δ_1 on (2.7) gives the relation (3.4) (coefficient of $1 \otimes x_k^2$ for $\hat{k} = 0$) and

$$t_i^k t_j^\ell (k|j\ell) - (-1)^{\hat{\imath}\hat{\jmath}+\hat{k}\hat{\ell}} t_j^\ell t_i^k (\ell|ik) = (-1)^{\hat{\imath}\hat{\jmath}} q^{-1} t_j^k t_i^\ell (k|i\ell) - (-1)^{\hat{k}\hat{\ell}} q\, t_i^\ell t_j^k (\ell|jk)$$
$$\text{for } i < j, k < \ell \tag{3.11}$$

which corresponds to the coefficient of $1 \otimes x_k x_\ell$ for $k < \ell$. The map δ_2 on (2.8) gives the following: the coefficient of $1 \otimes (y_k)^2$ gives the relation (3.1) for $\hat{\imath} = 0$ and $\hat{k} = 1$, while that of $1 \otimes y_k y_\ell$ gives (3.2). Finally, applying δ_2 to (2.9) gives (3.5) (coefficient of $1 \otimes (y_k)^2$ for $\hat{k} = 1$), while the coefficient of $1 \otimes y_k y_\ell$ gives

$$t_i^k t_j^\ell (k|j\ell) - (-1)^{\hat{\imath}\hat{\jmath}+\hat{k}\hat{\ell}} t_j^\ell t_i^k (\ell|ik) = (-1)^{\hat{k}\hat{\ell}} q^{-1} t_i^\ell t_j^k (\ell|jk) - (-1)^{\hat{\imath}\hat{\jmath}} q t_j^k t_i^\ell (k|i\ell)$$
$$\text{for } i < j, k < \ell. \tag{3.12}$$

For $q^2 \neq -1$, the set of relations (3.6) and (3.7) is equivalent to (3.11) and (3.12). □

We point out that one could alternatively define the algebra $M_q(n)$ through its coaction (δ, δ^*) on the pair $(\mathcal{A}_q, \mathcal{A}_q^*)$ of quantum spaces. One obtains the relations (3.1)–(3.7) by using the multiplication rule following from (3.9) in $M_q(n)\otimes\mathcal{A}_q$ but with the different rule $(t_i^j\otimes\xi_k)(t_\ell^m\otimes\xi_p) = (t_i^j t_\ell^m\otimes\xi_k\xi_p)(k|\ell m)^*$ in $M_q(n)\otimes\mathcal{A}_q^*$, where $(k|\ell m)^* = (k|\ell m)(-1)^{\hat\ell+\hat m}$.

We now turn to the question of the polynomiality of $M_q(n)$. Can we define a monomial basis for $M_q(n)$ and would this lead to constraints on the coefficients $(i|jk)$? At this point it is important to observe the similarities between the commutation relations of $M_q(n)$ and those of $E_q(n)$ given in (2.1). The difference lies only in the coefficients of these quadratic relations. For $(i|jk) = (-1)^{\hat\imath(\hat\jmath+\hat k)}$ they are in fact identical. It follows that the approach used by Manin [23] in his study of the structure of $E_q(n)$ can be applied to $M_q(n)$. Constraints on the three-index transposition coefficients will come from the examination of cubic monomials. *An ordering of the* t_i^j *is defined as follows:* $t_i^j < t_k^\ell$ *if either* $i > k$ *or* $i = k$ and $j > \ell$. A monomial in the t_i^j is normally ordered if, for any $t' < t''$ in this monomial, t' appears to the left of t'' and no odd t_i^j (i.e. with $\hat\imath + \hat\jmath = 1$) appear twice in the monomial.

Proposition 2. Normally ordered monomials in the t_i^j $(i, j = 1, \ldots, n)$ form a basis of $M_q(n)$ if the transposition coefficients satisfy

$$(j|\ell i)(j|km) = (j|ki)(j|\ell m). \tag{3.13}$$

Proof. We begin with quadratic monomials and refer the reader to [23, section 5.1] (the one-parameter case $q_{ij} = q$). Relations (3.1)–(3.7) for fixed i, j, k, ℓ are relations among elements of a 2×2 submatrix

$$T_s = \begin{pmatrix} t_i^k & t_i^\ell \\ t_j^k & t_j^\ell \end{pmatrix}$$

in T. It is obvious from these relations that $t_i^k t_i^\ell$, $t_i^k t_j^k$, $t_i^k t_j^\ell$, $t_i^\ell t_j^k$, $t_j^k t_j^\ell$ and $t_i^\ell t_j^\ell$ can be normally ordered without restrictions on the transposition coefficients. Now the ordering of any pair $t_a^b t_c^d$ can always be done from the appropriate submatrix T_s. It follows that normally ordered quadratic monomials form a basis of the quadratic part of $M_q(n)$ with no restrictions on the coefficients. Manin's ordering algorithm described in section 5.3 of [23] then applies and therefore normally ordered monomials span all of $M_q(n)$. Showing their independence is more tedious. One starts by considering the cubic part of $M_q(n)$. There are two distinct ways of reordering a monomial abc when $a > b > c$. We must check if these two paths lead to the same expressions. There are 22 cases one must consider (see sections 5.4 and 5.5 in [23] and theorem 3.3 in [29]). As an example consider the case $t_i^\ell t_j^p t_k^m$ with $i < j < k$ and $p < \ell < m$. Starting from the right

$$\begin{aligned} t_i^\ell(t_j^p t_k^m) &= (-1)^{\hat p\hat m+\hat\jmath\hat k+\hat m\hat\ell+\hat\imath\hat k+\hat\ell\hat p+\hat\imath\hat\jmath}(p|km)^{-1}(\ell|km)^{-1}(\ell|jp)^{-1} \\ &\quad\times(m|jp)(m|i\ell)(p|i\ell)t_k^m t_j^p t_i^\ell \\ &\quad+(-1)^{\hat\jmath\hat k+\hat\imath\hat k+\hat\imath\hat\jmath}(q^{-1}-q)(p|km)^{-1}(\ell|km)^{-1}(\ell|im)(p|im)t_k^\ell t_j^p t_i^m \\ &\quad+(-1)^{\hat\jmath\hat k+\hat\ell\hat p+\hat\imath\hat k+\hat m\hat\ell+\hat\imath\hat\jmath}(q^{-1}-q)(p|km)^{-1}(\ell|kp)^{-1}(\ell|jm)^{-1} \\ &\quad\times(p|jm)(p|i\ell)(m|i\ell)t_k^p t_j^m t_i^\ell \\ &\quad+(-1)^{\hat\jmath\hat k+\hat\ell\hat p+\hat\imath\hat k+\hat\imath\hat\jmath}(q^{-1}-q)^2(p|km)^{-1}(\ell|kp)^{-1}(\ell|jm)^{-1} \\ &\quad\times(p|jm)(p|i\ell)(\ell|im)t_k^p t_j^\ell t_i^m \end{aligned} \tag{3.14}$$

and starting from the left

$$
\begin{aligned}
(t_i^\ell t_j^p)t_k^m = {} & (-1)^{\hat\ell\hat p+\hat\imath\hat\jmath+\hat m\hat\ell+\hat\imath\hat k+\hat m\hat p+\hat\jmath\hat k}(\ell|jp)^{-1}(\ell|km)^{-1}(p|km)^{-1} \\
& \times (p|i\ell)(m|i\ell)(m|jp)t_k^m t_j^p t_i^\ell \\
& + (-1)^{\hat\imath\hat\jmath+\hat\imath\hat k+\hat\jmath\hat k}(q^{-1}-q)(\ell|km)^{-1}(p|k\ell)^{-1}(p|i\ell)(\ell|im)t_k^\ell t_j^p t_i^m \\
& + (-1)^{\hat\ell\hat p+\hat\imath\hat\jmath+\hat m\hat\ell+\hat k\hat\imath+\hat k\hat\jmath}(q^{-1}-q)(\ell|jp)^{-1}(\ell|km)^{-1}(p|km)^{-1} \\
& \times (p|i\ell)(m|i\ell)(p|jm)t_k^p t_j^m t_i^\ell \\
& + (-1)^{\hat\ell\hat p+\hat\imath\hat\jmath+\hat\imath\hat k+\hat\jmath\hat k}(q^{-1}-q)^2(\ell|jp)^{-1}(\ell|km)^{-1}(p|k\ell)^{-1} \\
& \times (p|i\ell)(\ell|im)(p|j\ell)t_k^p t_j^\ell t_i^m .
\end{aligned}
\tag{3.15}
$$

Comparing (3.14) and (3.15) we get the following restrictions for $i < j < k$ and $p < l < m$

$$
(p|im)(p|k\ell) = (p|i\ell)(p|km) \qquad (\ell|jp)(\ell|km) = (\ell|kp)(\ell|jm)
$$

$$
(p|j\ell)(p|km)(\ell|kp)(\ell|jm) = (p|jm)(\ell|km)(p|k\ell)(\ell|jp).
$$

These relations are obviously satisfied if (3.13) is satisfied. Similar results are obtained for the other cases. Therefore, normally ordered cubic monomials form a basis for the cubic part of $M_q(n)$. Application of the diamond lemma [30] then proves the proposition. □

Note that (3.13) restricts the coefficients $(i|jk)$ to be of the form $f_{ji}g_{ki}$, where f_{ij} and g_{ij} are arbitrary parameters. When one makes this restriction, the defining relations (3.8) of $M_q(n)$ can be written in a form reminiscent of (2.4)

$$
\check R T_1 P_f T_1 P_{g^t} = T_1 P_f T_1 P_{g^t}\check R \tag{3.16}
$$

where $T_1 = T \otimes I$, $(T)_i^j = t_i^j$, P_α is a generalized permutation operator with $(P_\alpha)_{ab}^{cd} = \delta_a^d\delta_b^c\alpha_{cd}$ and α^t is the transpose of α ($\alpha_{ij}^t \equiv \alpha_{ji}$).

In order to give $M_q(n)$ the structure of a bialgebra we must define two algebra homomorphisms $\Delta : M_q(n) \to M_q(n) \otimes M_q(n)$ and $\epsilon : M_q(n) \to k$ that satisfy the axioms

$$
\begin{aligned}
&\text{(compatibility)} && (m\otimes m)\cdot\Psi_{(23)}\cdot(\Delta\otimes\Delta) = \Delta\cdot m && (3.17)\\
&\text{(coassociativity)} && (\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta && (3.18)\\
&\text{(counit)} && (\epsilon\otimes\mathrm{id})\cdot\Delta = (\mathrm{id}\otimes\epsilon)\cdot\Delta = \mathrm{id} && (3.19)
\end{aligned}
$$

where m stands for the multiplication map $M_q(n)\otimes M_q(n) \to M_q(n)$.

The Ψ appearing in (3.17) is the transposition map $\Psi : M_q(n)\otimes M_q(n) \to M_q(n)\otimes M_q(n)$, which we choose to be of the simple form given by

$$
\Psi(t_i^j \otimes t_k^\ell) = (ij|k\ell)(t_k^\ell \otimes t_i^j) \tag{3.20}
$$

where the four-index transposition coefficients $(ij|kl)$ are for now arbitrary parameters. Without too much risk of confusion (we will use them only in the proofs of lemmas 1 and 2 and proposition 6) one may then also introduce more general transposition coefficients for arbitrary monomials a, b in $M_q(n)$ through

$$\Psi(a \otimes b) = (a|b)\, b \otimes a$$

and the relations (2.2) then reduce to the statements

$$(ab|c) = (a|c)(b|c) \qquad (a|bc) = (a|b)(a|c).$$

We assume henceforth that when acting on generators t_i^j the coproduct and counit retain the form

$$\Delta(t_i^j) = \sum_{k=1}^{n} t_i^k \otimes t_k^j \qquad \epsilon(t_i^j) = \delta_i^j \tag{3.21}$$

and first present some basic results regarding the action of the maps entering the above axioms on the algebra $\mathcal{A}$ freely generated by the t_i^j, subject only to the generalized statistics defined by the transposition map (3.20).

The compatibility axiom (3.17) simply makes natural the notation $\Delta(ab) = \Delta(a)\Delta(b)$ which enters the definition of 'homomorphism', given that the product in $M_q(n) \otimes M_q(n)$ is defined by the transposition map Ψ through (2.3).

Lemma 1. The maps $(\Delta \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes \Delta)\Delta$ are homomorphisms $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, provided the four-index transposition coefficients satisfy $(ij|mn) = (ik|mn)(kj|mn) = (ij|mp)(ij|pn)$.

Proof. Acting on the product of arbitrary monomials a, b in $\mathcal{A}$, one finds

$$\begin{aligned}(\mathrm{id} \otimes \Delta)\Delta(ab) &= a_{(1)}b_{(1)} \otimes \Delta(a_{(2)})\Delta(b_{(2)})\, (a_{(2)}|b_{(1)}) \\ &= a_{(1)}b_{(1)} \otimes a_{(2)(1)}b_{(2)(1)} \otimes a_{(2)(2)}b_{(2)(2)}\, (a_{(2)(2)}|b_{(2)(1)})(a_{(2)}|b_{(1)})\end{aligned}$$

where the convenient shorthand notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ is used. On the other hand

$$\begin{aligned}&(\mathrm{id} \otimes \Delta)\Delta(a) \cdot (\mathrm{id} \otimes \Delta)\Delta(b) \\ &\qquad = a_{(1)}b_{(1)} \otimes a_{(2)(1)}b_{(2)(1)} \otimes a_{(2)(2)}b_{(2)(2)}\, (a_{(2)(1)}|b_{(1)}) \\ &\qquad\quad \times (a_{(2)(2)}|b_{(1)})(a_{(2)(2)}|b_{(2)(1)}).\end{aligned}$$

The expressions are thus equivalent provided $(a_{(2)(1)}|b_{(1)})(a_{(2)(2)}|b_{(1)}) = (a_{(2)}|b_{(1)})$. Choosing, without loss of generality, a to be a single generator t_i^j and using the explicit form of the coproduct, one finds this holds if the four-index coefficients satisfy $(ij|mn) = (ik|mn)(kj|mn)$. The proof for $(\Delta \otimes \mathrm{id})\Delta$ is similar. □

Lemma 2. The maps $(\epsilon \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes \epsilon)\Delta$ are homomorphisms $\mathcal{A} \to \mathcal{A}$ provided $(ij|mm) = (ii|mn) = 1$.

Proof. Again acting on a product of arbitrary monomials

$$(\epsilon \otimes \mathrm{id})\Delta(ab) = \epsilon(a_{(1)})\epsilon(b_{(1)}) \otimes a_{(2)}b_{(2)}\, (a_{(2)}|b_{(1)})$$

which equals $(\epsilon \otimes \mathrm{id})\Delta(a) \cdot (\epsilon \otimes \mathrm{id})\Delta(b)$ provided $(a_{(2)}|b_{(1)}) = 1$ when $\epsilon(b_{(1)}) \neq 0$, which requires $(ij|mm) = 1$ for the given coproduct and counit. The proof for $(\mathrm{id} \otimes \epsilon)\Delta$ is similar. □

Having proven that the maps involved are homomorphisms on $\mathcal{A}$, we can easily verify the axioms (3.18) and (3.19) by acting on a single generator.

We next examine the compatibility of the given coproduct and counit with the quadratic relations (3.8) which restrict $\mathcal{A}$ to $M_q(n)$.

Proposition 3. If

$$(ij|k\ell) = (i|k\ell)(j|\ell m)(j|km)^{-1}$$

then the map $\Delta : M_q(n) \to M_q(n) \otimes M_q(n)$ is an algebra homomorphism.

Proof. Applying Δ on both sides of (3.8) we must show that

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} \Delta(t_i^k)\Delta(t_j^\ell)(k|j\ell) = \sum_{i,j=1}^{n} \Delta(t_a^i)\Delta(t_b^j)\check{R}^{k\ell}_{ij}(i|bj)$$

that is

$$\sum_{i,j,p,q=1}^{n} \check{R}^{ij}_{ab}(t_i^p t_j^q \otimes t_p^k t_q^\ell)(pk|jq)(k|j\ell) = \sum_{i,j,p,q=1}^{n} (t_a^p t_b^q \otimes t_p^i t_q^j)\check{R}^{k\ell}_{ij}(pi|bq)(i|bj).$$

Using $(pk|jq)(k|j\ell) = (p|jq)(k|q\ell)$ puts the left-hand side into a form on which (3.8) may be used directly. Similarly, the right-hand side may be thus transformed using (3.8). The resulting expressions are identical. □

Proposition 4. If $(i|jj) = (j|ii)$, then the counit $\epsilon : M_q(n) \to k$ is an algebra homomorphism.

Proof. Application of ϵ to the relations (3.8) leads straightforwardly to the requirement that

$$\check{R}^{k\ell}_{ab}(k|\ell\ell) = \check{R}^{k\ell}_{ab}(a|bb).$$

Use of the charge conservation of $\check{R}$ then reduces this to the given condition on $(i|jk)$. □

Combining the above results we obtain:

Theorem 1. The algebra $M_{q,\mu}(n)$ generated by t_i^j $(i, j = 1, \ldots, n)$ subject to the relations

$$\check{R}T_1 P_\mu T_1 P_\mu^{-1} = T_1 P_\mu T_1 P_\mu^{-1}\check{R} \tag{3.22}$$

is a bialgebra with the coproduct and counit given by

$$\Delta(t_i^j) = \sum_{k=1}^{n} t_i^k \otimes t_k^j \qquad \epsilon(t_i^j) = \delta_i^j$$

when the generalized statistics defined by the four-index transposition coefficients

$$(ij|kl) = (ij|kl)_\mu = \frac{\mu_{ki}\mu_{lj}}{\mu_{li}\mu_{kj}}$$

are used.

Proof. The solution of the conditions on the three- and four-index transposition coefficients obtained in lemmas 1 and 2, and propositions 2–4, is unique and equal to the following: $(i|jk) = (i|jk)_\mu = \mu_{ji}\mu_{ki}^{-1}$ and $(ij|kl) = (ij|kl)_\mu = \mu_{ki}\mu_{lj}\mu_{li}^{-1}\mu_{kj}^{-1}$ with the μ_{ab}'s $(a, b = 1, n)$ being arbitrary parameters. As a result the relations (3.8) or (3.16) take the form (3.22), upon use of the easily verified relation $P_{(\mu^{-1})^t} = P_\mu^{-1}$. □

We shall now give an interpretation of the parameters μ_{ij}. We will show how the maps (3.9) and (3.20), with $(i|k\ell) = (i|k\ell)_\mu$ and $(ij|k\ell) = (ij|k\ell)_\mu$, can be derived from a set of elementary maps $\Psi : Q_\alpha \otimes Q_\beta \to Q_\beta \otimes Q_\alpha$ $(\alpha, \beta = 1, 2)$ defined as

$$\begin{aligned} \Psi(x_i \otimes x_j) &= \mu_{ji}(x_j \otimes x_i) \qquad & \Psi(y_i \otimes y_j) &= \mu_{ji}(y_j \otimes y_i) \\ \Psi(x_i \otimes y_j) &= \mu_{ji}^{-1}(y_j \otimes x_i) \qquad & \Psi(y_i \otimes x_j) &= \mu_{ji}^{-1}(x_j \otimes y_i). \end{aligned} \tag{3.23}$$

Using (2.2a) and (3.23) one verifies that

$$\begin{aligned} \Psi(x_i \otimes (x_k \otimes y_\ell)) &= (i|k\ell)_\mu((x_k \otimes y_\ell) \otimes x_i) \\ \Psi(y_i \otimes (y_k \otimes x_\ell)) &= (i|k\ell)_\mu((y_k \otimes x_\ell) \otimes y_i). \end{aligned} \tag{3.24}$$

Comparison of the maps (3.24) and (3.9) suggest that when coacting on Q_1 (Q_2) the identification $t_k^\ell \equiv x_k \otimes y_\ell$ $(t_k^\ell \equiv y_k \otimes x_\ell)$ provides a realization of $M_{q,\mu}(n)$ in $Q_{1(2)} \otimes Q_{2(1)}$. This can be verified by making the above substitution for t_k^ℓ in the equations (3.1), (3.3), (3.4) and (3.11) ((3.1), (3.2), (3.5) and (3.12)) combined with the use of the maps (3.23) to define the product rule in $Q_\alpha \otimes Q_\beta$ and the defining relation of the quantum spaces Q_α. The maps (3.9) with $(i|k\ell) = (i|k\ell)_\mu$ are therefore consistent with the definition of the elementary maps given in (3.23). The same remark also applies to the maps (3.20) with $(ij|k\ell) = (ij|k\ell)_\mu$. Indeed, using (3.23) and (2.2a) one can verify that

$$\begin{aligned} \Psi((x_i \otimes y_j) \otimes (x_k \otimes y_\ell)) &= (ij|k\ell)_\mu((x_k \otimes y_\ell) \otimes (x_i \otimes y_j)) \\ \Psi((y_i \otimes x_j) \otimes (y_k \otimes x_\ell)) &= (ij|k\ell)_\mu((y_k \otimes x_\ell) \otimes (y_i \otimes x_j)). \end{aligned} \tag{3.25}$$

These transposition maps can be interpreted as realizations of the map (3.20) in $(Q_{1(2)} \otimes Q_{2(1)}) \otimes (Q_{1(2)} \otimes Q_{2(1)})$. Note that the second transposition is obvious since the coordinates x_i and y_i are treated on an equal footing in (3.23). As a consequence of the above considerations the free parameters μ_{ij} $(i, j = 1, n)$ may be interpreted as two-index transposition coefficients in $Q_\alpha \otimes Q_\beta$ $(\alpha, \beta = 1, 2)$ which serve as building blocks for the three- and four-index coefficients. The solutions $(i|k\ell)_\mu = \mu_{ki}\mu_{\ell i}^{-1}$ and $(ij|k\ell)_\mu = \mu_{ki}\mu_{\ell j}\mu_{\ell i}^{-1}\mu_{kj}^{-1}$ of theorem 1 are therefore consistent with the definition (3.23) for the μ_{ij}'s and the properties (2.2a) of braided maps.

Proposition 5. Representations $\rho^{(\pm)}$ of $M_{q,\mu}(n)$ are given by

$$\rho^{(\pm)}(t_i^j)_\alpha^\beta = (\check{R}^{\pm 1})_{\alpha i}^{j\beta}\mu_{\beta j}.$$

Proof. Applying $\rho^{(\pm)}$ to (3.22) we must prove that

$$\sum_{i,j,\nu=1}^{n} \check{R}^{ij}_{ab}(\check{R}^{\pm 1})^{k\nu}_{\alpha i}(\check{R}^{\pm 1})^{\ell\beta}_{\nu j}\mu_{jk}\mu^{-1}_{\ell k}\mu_{\nu k}\mu_{\beta\ell} = \sum_{i,j,\nu=1}^{n}(\check{R}^{\pm 1})^{i\nu}_{\alpha a}(\check{R}^{\pm 1})^{j\beta}_{\nu b}\check{R}^{k\ell}_{ij}\mu_{bi}\mu^{-1}_{ji}\mu_{\nu i}\mu_{\beta j}.$$

Note that in terms of its components the braid relations

$$(\check{R}^{\pm 1})_{12}(\check{R}^{\pm 1})_{23}\check{R}_{12} = \check{R}_{23}(\check{R}^{\pm 1})_{12}(\check{R}^{\pm 1})_{23}$$

are

$$\sum_{i,j,\nu=1}^{n} R^{ij}_{ab}(\check{R}^{\pm 1})^{k\nu}_{\alpha i}(\check{R}^{\pm 1})^{\ell\beta}_{\nu j} = \sum_{i,j,\nu=1}^{n}(\check{R}^{\pm 1})^{i\nu}_{\alpha a}(\check{R}^{\pm 1})^{j\beta}_{\nu b}\check{R}^{k\ell}_{ij}$$

and therefore $\rho^{(\pm)}$ will be a representation if, for any set (k,ℓ,β) and (a,b,α) of upper and lower indices, the products $\mu_{jk}\ \mu^{-1}_{\ell k}\ \mu_{\nu k}\ \mu_{\beta\ell}$ and $\mu_{bi}\ \mu^{-1}_{ji}\ \mu_{\nu i}\mu_{\beta j}$ are equal for all i, j and ν. In order to prove this we exploit the fact that $\check{R}$ satisfies charge conservation. It follows that only the cases $(k=b,\ \ell=\alpha,\ \beta=a)$, $(k=b,\ \ell=a,\ \beta=\alpha)$, $(k=\alpha,\ \ell=a,\ \beta=b)$, $(k=\alpha,\ \ell=b,\ \beta=a)$, $(k=a,\ \ell=\alpha,\ \beta=b)$ and $(k=a,\ \ell=b,\ \beta=\alpha)$ need to be considered. All other cases give $0=0$. For example, let us consider the case $(k=\alpha,\ \ell=b,\ \beta=a)$. The sum on the left-hand side contains two terms: a term corresponding to $(i=b,\ j=a,\ \nu=b)$ which gives $\mu_{jk}\ \mu^{-1}_{\ell k}\ \mu_{\nu k}\ \mu_{\beta\ell}=\mu_{a\alpha}\mu_{ab}$ and a term $(i=a,\ j=b,\ \nu=a)$ which gives $\mu_{jk}\mu^{-1}_{\ell k}\ \mu_{\nu k}\ \mu_{\beta\ell}=\mu_{a\alpha}\mu_{ab}$. The sum on the right-hand side contains only one term, namely $(i=\alpha,\ j=b,\ \nu=a)$ which gives $\mu_{bi}\ \mu^{-1}_{ji}\ \mu_{\nu i}\ \mu_{\beta j}=\mu_{a\alpha}\mu_{ab}$. Thus the products of μ's cancel and we are left with the braid relation. The μ's also cancel in the other five cases mentioned above. We will call $\rho^{(+)}$ and $\rho^{(-)}$ the fundamental and conjugate fundamental representations, respectively. □

Having proven that $M_{q,\mu}(n)$ is a bialgebra, the next step would be to seek to add a Hopf structure by defining an antipode S, assumed still to satisfy the defining axiom

$$m(S\otimes \mathrm{id})\Delta = m(\mathrm{id}\otimes S)\Delta = \nu\cdot\epsilon \tag{3.26}$$

where $\nu : k \to M_{q,\mu}(n)$ is the unit map. A suitable generalization of the property of being a graded antihomomorphism is that S should satisfy

$$m(S\otimes S)\Psi = S\cdot m \tag{3.27}$$

where here m stands for the multiplication map $m : M_{q,\mu}(n)\otimes M_{q,\mu}(n) \to M_{q,\mu}(n)$. Given the statistics defined by μ, this reads, e.g. for a product of generators, as

$$S(t^j_i t^\ell_k) = (ij|k\ell)_\mu\ S(t^\ell_k)S(t^j_i).$$

Since the coproduct is multiplicative on the generators of $M_{q,\mu}(n)$, it follows from application of (3.26) to a single generator that

$$S(T)T = TS(T) = \mathbf{1}. \tag{3.28}$$

Just as there is no antipode in Manin's $E_q(n)$, except in a formal sense, one also does not exist in $M_{q,\mu}(n)$. We next show that there are, however, no obstacles in principle to following Manin and defining a formal antipode in some universal extension of $M_{q,\mu}(n)$.

Proposition 6. If (3.28) holds, then the maps $m(S \otimes \mathrm{id})\Delta$, $m(\mathrm{id} \otimes S)\Delta$ and $\nu \cdot \epsilon$ are equivalent homomorphisms on $M_{q,\mu}(n)$.

Proof. For $m(S \otimes \mathrm{id})\Delta$ acting on a product of arbitrary elements a, b of $M_{q,\mu}(n)$, one sees that

$$\begin{aligned} m(S \otimes \mathrm{id})\Delta(ab) &= S(a_{(1)}b_{(1)})a_{(2)}b_{(2)}(a_{(2)}|b_{(1)})_\mu \\ &= S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}(a_{(1)}|b_{(1)})_\mu(a_{(2)}|b_{(1)})_\mu \\ &= S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}(a|b_{(1)})_\mu . \end{aligned}$$

Choosing, without loss of generality, a to be a generator t_i^j, equation (3.28) leads to $S(a_{(1)})a_{(2)}(a|b_{(1)})_\mu = \delta_i^j \mathbf{1} = S(a_{(1)})a_{(2)}$ since the four-index transposition coefficients satisfy $(ii|mn)_\mu = 1$. Thus one is led in general to

$$m(S \otimes \mathrm{id})\Delta(ab) = m(S \otimes \mathrm{id})\Delta(a) \cdot m(S \otimes \mathrm{id})\Delta(b).$$

Similar considerations apply to $m(\mathrm{id} \otimes S)\Delta$, and thus the equivalence of the two, and the general validity of (3.26), follow from (3.28). It only remains to prove that the action of S on the quadratic relations (3.8) imposes no further restrictions, i.e. that

$$\sum_{i,j=1}^{n} \check{R}_{ab}^{ij} S(t_j^\ell) S(t_i^k)(ik|j\ell)_\mu(k|j\ell)_\mu = \sum_{i,j=1}^{n} \check{R}_{ij}^{k\ell} S(t_b^j) S(t_a^i)(ai|bj)_\mu(i|bj)_\mu .$$

Multiplying on both sides by $(a|qb)_\mu t_p^a t_q^b$ from the left and by $(m|lu)_\mu t_k^m t_l^u$ from the right, leads through application of (3.28) back to (3.8) itself. □

We have shown that $M_{q,\mu}(n)$ coacts as an algebra on the pair $(Q_1(n), Q_2(n))$ of quantum spaces through the homomorphisms δ_1, δ_2. It is natural to inquire further whether the coactions give corepresentations of $M_{q,\mu}(n)$ as a bialgebra, that is, whether the maps δ_1, δ_2 satisfy the coassociativity and counit axioms which would make $Q_1(n)$ and $Q_2(n)$ $M_{q,\mu}(n)$-comodules

$$(\Delta \otimes \mathrm{id})\delta_i = (\mathrm{id} \otimes \delta_i)\delta_i \qquad Q_i \to M_q \otimes M_q \otimes Q_i \tag{3.29}$$

$$(\epsilon \otimes \mathrm{id})\delta_i = \mathrm{id} \qquad Q_i \to k \otimes Q_i = Q_i . \tag{3.30}$$

One may show that all the mappings involved are homomorphisms, and that the axioms do indeed hold for $M_{q,\mu}(n)$.

On the other hand, one may approach the entire problem of defining the algebraic and coalgebraic structure of the coacting object, as well as any restrictions on the statistics to be used, by starting from given coactions on quantum spaces which are assumed to be comodule algebras.

Indeed, given coactions δ_i of the form (3.10) on, for now, freely generated algebras $Q_1(n)$ and $Q_2(n)$, and statistics defined through three- and four-index transposition coefficients as in (3.9) and (3.20), one finds that the axioms (3.29) and (3.30) applied to a single x_i or y_i then define the coproduct and counit to act as (3.21) on a single generator t_i^j, and one can prove:

Lemma 3.
(a) If the three- and four-index transposition coefficients are related by $(ij|mn) = (i|mn)(j|mn)^{-1}$, then $(\mathrm{id} \otimes \delta_i)\delta_{\bar{i}}$ are homomorphisms.
(b) If the three-index transposition coefficients satisfy $(i|jk) = (i|jm)(i|mk)$, then the coproduct Δ defined by (3.29) is a homomorphism.
(c) If the three-index transposition coefficients satisfy $(i|jj) = 1$, then the counit ϵ defined by (3.30) is a homomorphism.

Proof. The proofs proceed on exactly the same lines as those of lemmas 1 and 2. □

Furthermore, proposition 1 may now be seen as deriving the relations (3.8) in the algebra $M_q(n)$ from the demand that the imposition of the quadratic relations (2.6)–(2.9) be compatible with the form (3.10) of the homomorphisms δ_i.

The above conditions again combine to give the transposition coefficients defined by parameters μ_{ij}, and one may summarize the results as:

Theorem 2. The quantum spaces $Q_1(n)$, $Q_2(n)$ of (2.6)–(2.9), are $M_{q,\mu}(n)$-comodule algebras.

The point here is that the particular statistics, and a consistent coalgebraic structure, follow once one has chosen the quantum spaces and coactions, and assumed statistics of the general form given by (3.9) and (3.20). There is, for example, no further need to check that the Δ defined by (3.29) preserves the relations (3.22).

We conclude this section by connecting with Majid's work on braided groups [21]. The transposition maps (3.9), (3.20) and (3.23) can be expressed in terms of P_μ as

$$\Psi(x_i \otimes x_j) = \sum_{\alpha,\beta} (P_\mu)_{ij}^{\alpha\beta} (x_\alpha \otimes x_\beta) \qquad \Psi(x_i \otimes y_j) = \sum_{\alpha,\beta} (\tilde{P}_\mu)_{ij}^{\alpha\beta} (x_\alpha \otimes y_\beta)$$

$$\Psi(c_i \otimes t_k^\ell) = \sum_{\substack{\alpha_1,\alpha_2\\ \beta_1,\beta_2}} (P_\mu)_{ik}^{\alpha_1\beta_1} (\tilde{P}_\mu)_{\beta_1\ell}^{\alpha_2\beta_2} (t_{\alpha_1}^{\alpha_2} \otimes c_{\beta_2}) \tag{3.31}$$

$$\Psi(t_i^j \otimes t_k^\ell) = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3,\beta_4}} (\tilde{P}_\mu)_{jk}^{\alpha_1\beta_1} (P_\mu)_{\beta_1\ell}^{\alpha_2\beta_2} (P_\mu)_{i\alpha_1}^{\alpha_3\beta_3} (\tilde{P}_\mu)_{\beta_3\alpha_2}^{\alpha_4\beta_4} (t_{\alpha_3}^{\alpha_4} \otimes t_{\beta_4}^{\beta_2})$$

where $\tilde{P}_\mu \equiv [(P_\mu^{\mathrm{t}})^{-1}]^{\mathrm{t}}$, that is $(\tilde{P}_\mu)_{ab}^{cd} = \delta_a^d \delta_b^c \mu_{cd}^{-1}$. P_μ is therefore the braiding matrix of $M_{q,\mu}(n)$. These maps are special cases, with the braiding matrix set to P_μ, of the more general (with arbitrary R-matrix) transposition maps used by Majid to define a structure somewhat similar to that obtained from the FRT formalism, but which exhibits total covariance under the action of a bialgebra A associated with an R-matrix. In his approach, the same R-matrix is used to define the algebraic relations as well as to define Ψ. Our construction can be viewed as a hybrid, with the algebraic relations being defined by the R-matrix given in (2.5a), in the framework of the braided tensor category defined in terms of another R-matrix, here the generalized permutation operator P_μ. The distinct role of these two elements is especially clear in the comodule algebra approach, where one R-matrix defines the quantum spaces [28] while the other defines the statistics as in (3.31). One might consider transposition maps that are defined in terms of more general R-matrices. It appears [31] that the possibilities are few.

4. Dual space and P_μ-statistics

Consider elements $\mathbf{1}'$ and $L^{(\pm)j}_{\ i}$ $(i, j = 1, \ldots, n)$ in the dual space of the algebra $M_{q,\mu}(n)$, which are defined by the dual pairings

$$\begin{aligned} &\langle \mathbf{1}' | T_1 T_2 \cdots T_k \rangle = I^{\otimes k} && \langle \mathbf{1}' | \mathbf{1} \rangle = 1 \\ &\langle L^{(\pm)} | T_1 T_2 \cdots T_k \rangle = R_1^{(\pm)} R_2^{(\pm)} \cdots R_k^{(\pm)} && \langle L^{(\pm)} | \mathbf{1} \rangle = I \end{aligned} \tag{4.1}$$

where $L^{(\pm)} = (L^{(\pm)j}_{\ i})$ and $I = (\delta_i^j)$ are $n \times n$ matrices, and $T_i = I \otimes I \otimes \cdots \otimes T \otimes \cdots I$ (T in the ith position of a tensor product of k matrices). $R_i^{(\pm)}$ acts on factors number 0 (corresponding to the $L^{(\pm)}$-space) and i, and coincides there with $R^{(\pm)} = \check{R}^{\pm 1} P_\mu$. It follows from these duality conditions that $L^{(+)}$ and $L^{(-)}$ are upper and lower triangular matrices, respectively. In order to prove the consistency of (4.1) with relations (3.22) we must show that

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} \langle L^{(\pm)\beta}_{\ \alpha} | t^k_i t^\ell_j \rangle \mu_{jk} \mu^{-1}_{\ell k} = \sum_{i,j=1}^{n} \langle L^{(\pm)\beta}_{\ \alpha} | t^i_a t^j_b \rangle \check{R}^{k\ell}_{ij} \mu_{bi} \mu^{-1}_{ji}. \tag{4.2}$$

However, equation (4.1) gives precisely the representation

$$\langle L^{(\pm)} | T_1 T_2 \ldots T_k \rangle = \rho^{(\pm)}(T_1 T_2 \ldots T_k)$$

so proposition 5 shows that (4.2) holds. In what follows we use the notation $\mu^{\mathrm{t}}_{ij} = \mu_{ji}$, $[P_{\mu^{\mathrm{t}}}]^{cd}_{ab} \equiv \delta^d_a \delta^c_b \mu_{dc}$, $(a|bc)_{\mu^{\mathrm{t}}} = \mu_{ab}\mu^{-1}_{ac}$ and $(ij|k\ell)_{\mu^{\mathrm{t}}} = (i|k\ell)_{\mu^{\mathrm{t}}}(j|\ell k)_{\mu^{\mathrm{t}}}$.

Theorem 3. The algebra $U_{q,\mu}(n)$ generated by $L^{(\pm)j}_{\ i}$ $(i, j = 1, \ldots, n)$ and the unit element $\mathbf{1}'$, subject to the quadratic relations

$$\check{\check{R}} L_1^{(\epsilon)} P_{\mu^{\mathrm{t}}} L_1^{(\epsilon')} P^{-1}_{\mu^{\mathrm{t}}} = L_1^{(\epsilon')} P_{\mu^{\mathrm{t}}} L_1^{(\epsilon)} P^{-1}_{\mu^{\mathrm{t}}} \check{\check{R}} \tag{4.3}$$

(where $\check{\check{R}}^{ij}_{ab} = \check{R}^{ji}_{ba}$ and $(\epsilon, \epsilon') = (\pm, \pm)$ or $(+, -)$), and equipped with a coproduct $\Delta : U_{q,\mu}(n) \to U_{q,\mu}(n) \otimes U_{q,\mu}(n)$ and counit $\epsilon : U_{q,\mu}(n) \to k$ defined by

$$\Delta(L^{(\pm)j}_{\ i}) = \sum_{k=1}^{n} L^{(\pm)k}_{\ i} \otimes L^{(\pm)j}_{\ k} \qquad \epsilon(L^{(\pm)j}_{\ i}) = \delta^j_i$$

with statistics defined by the transposition map $\Psi : U_{q,\mu}(n) \otimes U_{q,\mu}(n) \to U_{q,\mu}(n) \otimes U_{q,\mu}(n)$ given by

$$\Psi(L^{(\epsilon)j}_{\ i} \otimes L^{(\epsilon')\ell}_{\ k}) = (ij|k\ell)_{\mu^{\mathrm{t}}} \ (L^{(\epsilon')\ell}_{\ k} \otimes L^{(\epsilon)j}_{\ i}) \tag{4.4}$$

is a bialgebra dual to $M_{q,\mu}(n)$.

Proof. Duality of bialgebras requires the duality of the multiplication and coproduct maps in the sense

$$\langle xy|a\rangle = \langle x \otimes y|\Delta(a)\rangle \qquad \langle x|ab\rangle = \langle \Delta(x)|a \otimes b\rangle$$

which defines the dual pairing between all elements of $U_{q,\mu}(n)$ and $M_{q,\mu}(n)$ given $\langle L^{(\pm)b}{}_a|t_i^j\rangle = R^{(\pm)bj}{}_{ai}$, $\langle 1'|t_i^j\rangle = \delta_i^j$ and $\langle L^{(\pm)b}{}_a|1\rangle = \delta_a^b$. This agrees with the pairings already given in (4.1), for the given form of coproduct in $U_{q,\mu}(n)$. Since the four-index transposition coefficient $(ij|k\ell)_{\mu^t}$ is simply the transpose $(\mu^t_{ij} \to \mu_{ji})$ of $(ij|k\ell)_\mu$, it follows that the maps Δ and ϵ satisfy the axioms (3.17)–(3.19).

Relations (4.3) may be written in terms of components as

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} L^{(\epsilon)k}{}_j L^{(\epsilon')\ell}{}_i (k|i\ell)_{\mu^t} = \sum_{i,j=1}^{n} L^{(\epsilon')i}{}_b L^{(\epsilon)j}{}_a \check{R}^{\ell k}_{ji} (i|aj)_{\mu^t}. \tag{4.5}$$

We must show that, for example

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} \langle L^{(+)k}{}_j L^{(-)\ell}{}_i | t^{u_1}_{m_1} \cdots t^{u_p}_{m_p}\rangle \mu_{ki}\mu^{-1}_{k\ell} = \sum_{i,j=1}^{n} \langle L^{(-)i}{}_b L^{(+)j}{}_a | t^{\mu_1}_{m_1} \cdots t^{u_p}_{m_p}\rangle \check{R}^{\ell k}_{ji} \mu_{ia}\mu^{-1}_{ij}.$$

In the case $p = 1$ we must prove that

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} \langle L^{(+)k}{}_j \otimes L^{(-)\ell}{}_i | \Delta(t^{u_1}_{m_1})\rangle \mu_{ki}\mu^{-1}_{k\ell} = \sum_{i,j=1}^{n} \langle L^{(-)i}{}_b \otimes L^{(+)j}{}_a | \Delta(t^{u_1}_{m_1})\rangle \check{R}^{\ell k}_{ji} \mu_{ia}\mu^{-1}_{ij}$$

which reads

$$\sum_{p,i,j=1}^{n} \check{R}^{ij}_{ab} \check{R}^{pk}_{jm_1} (\check{R}^{-1})^{u_1\ell}_{ip} \mu_{kp}\mu_{\ell u_1}\mu_{ki}\mu^{-1}_{k\ell} = \sum_{p,i,j=1}^{n} (\check{R}^{-1})^{pi}_{bm_1} \check{R}^{u_1 j}_{ap} \check{R}^{\ell k}_{ji} \mu_{ip}\mu_{ju_1}\mu_{ia}\mu^{-1}_{ij}. \tag{4.6}$$

Using the braid relation

$$\check{R}_{12}\check{R}_{23}(\check{R}^{-1})_{12} = (\check{R}^{-1})_{23}\check{R}_{12}\check{R}_{23}$$

one proves (4.6) in the same way as in proposition 5, by showing that for any given set (k, ℓ, u_1) and (a, b, m_1) the products $\mu_{kp}\mu_{\ell u_1}\mu_{ki}\mu^{-1}_{k\ell}$ and $\mu_{ip}\mu_{ju_1}\mu_{ia}\mu^{-1}_{ij}$ are identical constants within each summation. For $p = 2$ we must show that

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab} \langle \Delta(L^{(+)k}{}_j L^{(-)\ell}{}_i) | t^{u_1}_{m_1} \otimes t^{u_2}_{m_2}\rangle \mu_{ki}\mu^{-1}_{k\ell}$$

$$= \sum_{i,j=1}^{n} \langle \Delta(L^{(-)i}{}_b L^{(+)j}{}_a) | t^{\mu_1}_{m_1} \otimes t^{u_2}_{m_2}\rangle \check{R}^{\ell k}_{ji} \mu_{ia}\mu^{-1}_{ij} \tag{4.7}$$

but

$$\Delta(L^{(\pm)b_1}{}_{a_1} L^{(\mp)b_2}{}_{a_2}) = \sum_{\alpha_1,\alpha_2=1}^{n} (L^{(\pm)\alpha_1}{}_{a_1} L^{(\mp)\alpha_2}{}_{a_2} \otimes L^{(\pm)b_1}{}_{\alpha_1} L^{(\mp)b_2}{}_{\alpha_2})(\alpha_1 b_1|a_2\alpha_2)_{\mu^t}$$

so (4.7) reads as

$$\sum_{i,j,\alpha_1,\alpha_2=1}^{n} \check{R}^{ij}_{ab}\langle L^{(+)\alpha_1}{}_{j} L^{(-)\alpha_2}{}_{i}|t^{u_1}_{m_1}\rangle\langle L^{(+)k}{}_{\alpha_1} L^{(-)\ell}{}_{\alpha_2}|t^{u_2}_{m_2}\rangle u_{\alpha_1 i} u^{-1}_{\alpha_1\alpha_2}\mu_{k\alpha_2}\mu^{-1}_{k\ell}$$
$$= \sum_{i,j,\alpha_1,\alpha_2=1}^{n} \langle L^{(-)\alpha_1}{}_{b} L^{(+)\alpha_2}{}_{a}|t^{u_1}_{m_1}\rangle\langle L^{(-)i}{}_{\alpha_1} L^{(+)j}{}_{\alpha_2}|t^{u_2}_{m_2}\rangle \check{R}^{\ell k}_{ji}\mu_{\alpha_1 a}\mu^{-1}_{\alpha_1\alpha_2}\mu_{i\alpha_2}\mu^{-1}_{ij}. \tag{4.8}$$

We prove (4.8) by transforming its left-hand side into its right-hand side using the results of the $p = 1$ case. We have just proved that

$$\sum_{i,j=1}^{n} \check{R}^{ij}_{ab}\langle L^{(+)\alpha_1}{}_{j} L^{(-)\alpha_2}{}_{i}|t^{u_1}_{m_1}\rangle\mu_{\alpha_1 i}\mu^{-1}_{\alpha_1\alpha_2} = \sum_{i,j=1}^{n}\langle L^{(-)i}{}_{b} L^{(+)j}{}_{a}|t^{u_1}_{m_1}\rangle\mu_{ia}\mu^{-1}_{ij}\check{R}^{\alpha_2\alpha_1}_{ji}$$

so the left-hand side of (4.8) is equal to

$$\sum_{i,j,\alpha_1,\alpha_2=1}^{n} \langle L^{(-)i}{}_{b} L^{(+)j}{}_{a}|t^{u_1}_{m_1}\rangle \check{R}^{\alpha_2\alpha_1}_{ji}\langle L^{(+)k}{}_{\alpha_1} L^{(-)\ell}{}_{\alpha_2}|t^{u_2}_{m_2}\rangle\mu_{k\alpha_2}\mu^{-1}_{k\ell}\mu_{ia}\mu^{-1}_{ij}. \tag{4.9}$$

From the $p = 1$ case we have that

$$\sum_{\alpha_1,\alpha_2=1}^{n} \check{R}^{\alpha_2\alpha_1}_{ji}\langle L^{(+)k}{}_{\alpha_1} L^{(-)\ell}{}_{\alpha_2}|t^{u_2}_{m_2}\rangle\mu_{k\alpha_2}\mu^{-1}_{k\ell} = \sum_{\alpha_1,\alpha_2=1}^{n}\langle L^{(-)\alpha_1}{}_{i} L^{(+)\alpha_2}{}_{j}|t^{u_2}_{m_2}\rangle\mu_{\alpha_1 j}\mu^{-1}_{\alpha_1\alpha_2}\check{R}^{\ell k}_{\alpha_2\alpha_1}. \tag{4.10}$$

Substituting (4.10) into (4.9) and with the appropriate change of variables ($i \to \alpha_1$, $j \to \alpha_2$, $\alpha_1 \to i$ and $\alpha_2 \to j$) one obtains the right-hand side of (4.8), thereby proving the $p = 2$ case. Note that from the coassociativity of the coproduct it follows that

$$\langle\Delta(L^{b_1}_{a_1}L^{b_2}_{a_2})|t^{u_1}_{m_1}\otimes t^{u_2}_{m_2}t^{u_3}_{m_3}\cdots t^{u_p}_{m_p}\rangle = \langle\Delta(L^{b_1}_{a_1}L^{b_2}_{a_2})|t^{u_1}_{m_1}t^{u_2}_{m_2}\otimes t^{u_3}_{m_3}\cdots t^{u_p}_{m_p}\rangle$$
$$= \cdots \langle\Delta(L^{b_1}_{a_1}L^{b_2}_{a_2})|t^{u_1}_{m_1}t^{u_2}_{m_2}\cdots\otimes t^{u_p}_{m_p}\rangle. \tag{4.11}$$

In order to prove the $p = 3$ case we must therefore show that

$$\sum_{i,j,\alpha_1,\alpha_2=1}^{n} \check{R}^{ij}_{ab}\langle L^{(+)\alpha_1}{}_{j} L^{(-)\alpha_2}{}_{i}|t^{u_1}_{m_1}t^{u_2}_{m_2}\rangle\langle L^{(+)k}{}_{\alpha_1} L^{(-)\ell}{}_{\alpha_2}|t^{u_3}_{m_3}\rangle\mu_{\alpha_1 i}\mu^{-1}_{\alpha_1\alpha_2}\mu_{k\alpha_2}\mu^{-1}_{k\ell}$$
$$= \sum_{i,j,\alpha_1,\alpha_2=1}^{n} \langle L^{(-)\alpha_1}{}_{b} L^{(+)\alpha_2}{}_{a}|t^{u_1}_{m_1}t^{\mu_2}_{m_2}\rangle\langle L^{(-)i}{}_{\alpha_1} L^{(+)j}{}_{\alpha_2}|t^{u_3}_{m_3}\rangle$$
$$\times \check{R}^{\ell k}_{ji}\mu_{\alpha_1 a}\mu^{-1}_{\alpha_1\alpha_2}\mu_{i\alpha_2}\mu^{-1}_{ij}. \tag{4.12}$$

The proof proceeds in the same way as for the $p = 2$ case. One first transforms the first and second brackets $\langle\,|\,\rangle$ on the left-hand side of (4.12) into the corresponding ones on the right-hand side using the $p = 2$ and $p = 1$ results. The generalization to arbitrary values of p is obvious. The cases $(\epsilon, \epsilon') = (\pm, \pm)$ are proven in exactly the same way. The proofs that, with the statistics defined by (4.4), the maps Δ and ϵ are algebraic homomorphisms of $U_{q,\mu}(n)$ proceed just as those for the coproduct and counit in $M_{q,\mu}(n)$, but with transposition coefficients everywhere defined by μ^{t}, since this is what appears in (4.5). $\square$

In proving the above, one has proven that the dual pairing also gives a representation ρ of $U_{q,\mu}(n)$ through

$$\rho(L^{(\pm)j}{}_i)^\beta_\alpha = \langle L^{(\pm)j}{}_i | t^\beta_\alpha \rangle = (\check{R}^{\pm 1})^{\beta j}_{i\alpha}\mu_{j\beta}.$$

We stress the appearance of 'transposed' statistics (4.4) in the dual space $U_{q,\mu}(n)$, and note that, as a consequence, an antipode map S defined on $U_{q,\mu}(n)$ would, for example, have to satisfy

$$S(L^{(\epsilon)j}{}_i L^{(\epsilon')\ell}{}_k) = (ij|kl)_{\mu^t} S(L^{(\epsilon')\ell}{}_k) S(L^{(\epsilon)j}{}_i).$$

5. A two-dimensional example

It is well known that the one-parameter deformations $GL_q(1|1)$ and $U_q(g\ell(1|1))$ of the general linear supergroup $GL(1|1)$ and its enveloping algebra $U(g\ell(1|1))$ are associated with the following solution of the braid relation:

$$\check{R} = \begin{pmatrix} 1 & & & \\ & (1-q^2) & q & \\ & q & 0 & \\ & & & -q^2 \end{pmatrix}. \tag{5.1}$$

Also related to this R-matrix are the Hopf algebras $X_q(2)$ and its dual $\hat{X}_q(2)$, proposed in [16]. Although the structure of these algebras resembles that of $U_q(g\ell(1|1))$ and $GL(1|1)$, they are not deformations of $U(g\ell(1|1))$ and $GL(1|1)$ in that these are not recovered when $q \to 1$. So there seem to be two distinct pairs of quantum algebras associated with the same R-matrix. There has been some confusion on this issue. One interpretation [8, 10] is that $X_q(2)$ and its dual are the algebras one obtains by using the FRT formalism without treating the gradings properly. In other words, with the R-matrix (5.1) one should use the graded version of the FRT formalism. This question was examined in [9, 32]. In the case of $U_q(g\ell(1|1))$ and $X_q(2)$ it was shown that they are both generated by four generators H, $\psi^\pm$ and X_0 which satisfy the relations

$$\begin{aligned} &[H, \psi^\pm] = \pm 2\psi^\pm \qquad && [H, X_0] = [X_0, \psi^\pm] = 0 \\ &(\psi^\pm)^2 = 0 && \psi^+\psi^- + \psi^-\psi^+ = \frac{q^{2X_0} - 1}{q^2 - 1} \end{aligned} \tag{5.2}$$

which are, of course, the familiar supersymmetric relations. So as algebras they are isomorphic, but it was found that the difference lies in the statistics they obey, which manifests itself at the level of the Hopf structure. For $U_q(g\ell(1|1))$ the coproduct and antipode acting on $\psi^\pm$ give

$$\Delta(\psi^\pm) = 1 \otimes \psi^\pm + \psi^\pm \otimes q^{X_0} \qquad S(\psi^\pm) = -\psi^\pm q^{-X_0}. \tag{5.3}$$

A Z_2 grading is associated with this algebra, with $\hat{\psi}^\pm = 1$ and $\hat{H} = \hat{X}_0 = 0$. This grading governs the product rule in $U_q \otimes U_q$ as well as the antipode of a product in U_q. On the other hand, the coproduct and antipode for $X_q(2)$ satisfy

$$\Delta(\psi^\pm) = \mathrm{i}^{\pm(H-X_0)} \otimes \psi^\pm + \psi^\pm \otimes q^{X_0} \qquad S(\psi^\pm) = \psi^\pm \mathrm{i}^{\mp(H-X_0)} q^{-X_0}. \tag{5.4}$$

For this algebra the statistics are purely bosonic in the sense that the transposition map, Ψ, is simply $\Psi(a \otimes b) = b \otimes a$ for $a, b \in X_q(2)$. An independent study was made by Majid and Rodriguez-Plaza [11] in which they find similar results and show that these two structures are related by 'superization'. The purpose of this section is to show that these two pairs of quantum algebras are in fact only special cases of a more general structure with P_μ-statistics, and may indeed be viewed as just two points in a continuous family of such structures.

Let $M_{q,\mu}(0,1)$ be the bialgebra $M_{q,\mu}(2)$ of theorem 1 with format $(0,1)$ made explicit. From (3.22), the commutation relations are

$$\begin{aligned}
&(t_1^2)^2 = 0 && (t_2^1)^2 = 0 \\
&t_1^1 t_1^2 - \mu_{11}^{-1}\mu_{21} q^{-1} t_1^2 t_1^1 = 0 && t_2^1 t_2^2 + q t_2^2 t_2^1 \mu_{22}\mu_{12}^{-1} = 0 \\
&t_1^1 t_2^1 - q^{-1}\mu_{11}\mu_{21}^{-1} t_2^1 t_1^1 = 0 && t_1^2 t_2^2 + q\mu_{12}\mu_{22}^{-1} t_2^2 t_1^2 = 0 \\
&t_1^1 t_2^2 - t_2^2 t_1^1 = (q^{-1} - q)\mu_{11}\mu_{21}^{-1} t_2^1 t_1^2 && t_1^2 t_2^1 = \mu_{12}\mu_{11}\mu_{22}^{-1}\mu_{21}^{-1} t_2^1 t_1^2.
\end{aligned}$$

Note that for $\mu_{ij} = (-1)^{\hat{i}\hat{j}}$ these relations reduce to those of $GL_q(1|1)$ while for $\mu_{ij} = 1$ we get those of $\hat{X}_q(2)$ [16]. The coproduct and counit are those given in theorem 1. We may also extend this bialgebra and define a Hopf algebra $M^+_{q,\mu}(0,1)$ by adding inverses $(t_1^1)^{-1}$ and $(t_2^2)^{-1}$ of t_1^1 and t_2^2 in order to define an antipode through

$$\begin{aligned}
&S(t_1^1) = (t_1^1)^{-1} + (t_1^1)^{-1} t_1^2 (t_2^2)^{-1} t_2^1 (t_1^1)^{-1} && S(t_2^2) = (t_2^2)^{-1} + (t_2^2)^{-1} t_2^1 (t_1^1)^{-1} t_1^2 (t_2^2)^{-1} \\
&S(t_1^2) = -(t_1^1)^{-1} t_1^2 (t_2^2)^{-1} && S(t_2^1) = -(t_2^2)^{-1} t_2^1 (t_1^1)^{-1} \\
&S((t_1^1)^{-1}) = t_1^1 - t_1^2 (t_2^2)^{-1} t_2^1 && S((t_2^2)^{-1}) = t_2^2 - t_2^1 (t_1^1)^{-1} t_1^2.
\end{aligned} \tag{5.5}$$

The algebraic relations and counit in $M^+_{q,\mu}(0,1)$ are natural ones, and the coproduct satisfies

$$\begin{aligned}
\Delta((t_1^1)^{-1}) &= (t_1^1)^{-1} \otimes (t_1^1)^{-1} - (t_1^1)^{-1} t_1^2 (t_1^1)^{-1} \otimes (t_1^1)^{-1} t_2^1 (t_1^1)^{-1} \\
\Delta((t_2^2)^{-1}) &= (t_2^2)^{-1} \otimes (t_2^2)^{-1} - (t_2^2)^{-1} t_2^1 (t_2^2)^{-1} \otimes (t_2^2)^{-1} t_1^2 (t_2^2)^{-1}.
\end{aligned}$$

It is easily verified that the map defined in (5.5) satisfies (3.28). Note that in form these mappings are identical to those of the supersymmetric case given in [6], and also correspond to those given in [16]. We stress that the P_μ-statistics must be used.

We now turn to the dual space of $M_{q,\mu}(0,1)$ and consider $U_{q,\mu}(0,1)$, whose commutation relations are

$$\begin{aligned}
&(L^{(+)2}{}_1)^2 = 0 \qquad (L^{(-)1}{}_2)^2 = 0 \\
&[L^{(\epsilon)i}{}_i, L^{(\epsilon')k}{}_k] = 0 \\
&(L^{(\pm)2}{}_2)(L^{(-)1}{}_2) = -q^{\pm 1} x (L^{(-)1}{}_2)(L^{(\pm)2}{}_2) \\
&(L^{(\pm)2}{}_2)(L^{(+)2}{}_1) = -q^{\pm 1} x^{-1} (L^{(+)2}{}_1)(L^{(\pm)2}{}_2) \\
&(L^{(\pm)1}{}_1)(L^{(-)1}{}_2) = q^{\mp 1} y^{-1} (L^{(-)1}{}_2)(L^{(\pm)1}{}_1) \\
&(L^{(\pm)1}{}_1)(L^{(+)2}{}_1) = q^{\pm 1} y (L^{(+)2}{}_1)(L^{(\pm)1}{}_1) \\
&(L^{(+)2}{}_1)(L^{(-)1}{}_2) - y^{-1} x (L^{(-)1}{}_2)(L^{(+)2}{}_1) = (q^{-1} - q) x [L^{(-)2}{}_2)(L^{(+)1}{}_1) - L^{(+)2}{}_2)(L^{(-)1}{}_1)]
\end{aligned}$$

where $x \equiv \mu_{21}\mu_{22}^{-1}$ and $y \equiv \mu_{12}\mu_{11}^{-1}$. We now specialize and consider the case $\mu_{ij} = t^{\hat{i}\hat{j}}$, where t is an arbitrary parameter (a more general treatment will be reported elsewhere). The resulting algebra $U_{(q,t)}(0,1)$, dual to $M_{(q,t)}(0,1)$, may then be realized in terms of generators H, $\psi^{\pm}$ and X_0 which satisfy the relations (5.2), by making the identifications (where i stands for the complex number $\mathrm{i}^2 = -1$)

$$L^{(\pm)1}{}_1 = q^{\pm(H-X_0)/2} \qquad L^{(\pm)2}{}_2 = q^{\pm(H+X_0)/2}t^{(H-X_0)/2}\mathrm{i}^{-(H-X_0)}$$

$$L^{(+)2}{}_1 = (1-q^2)\psi^{+}q^{(H-X_0)/2}\mathrm{i}^{-(H-X_0)}t^{(H-X_0)/2}$$

$$L^{(-)1}{}_2 = -(1-q^2)\psi^{-}q^{-(H+X_0)/2}.$$

In this form, the introduction of inverse elements and the passage to $U^{+}_{(q,t)}(0,1)$ is transparent. The coproduct and antipode on $\psi^{\pm}$ take the forms

$$\Delta(\psi^{\pm}) = \mathrm{i}^{\pm(H-X_0)}t^{\mp(H-X_0)/2} \otimes \psi^{\pm} + \psi^{\pm} \otimes q^{X_0}$$

$$S(\psi^{\pm}) = t\psi^{\pm}\mathrm{i}^{\mp(H-X_0)}t^{\pm(H-X_0)/2}q^{-X_0}.$$

In summary, $U^{+}_{(q,t)}(0,1)$ is a Hopf algebra with two types of deformation parameters. The parameter q deforms the quadratic relations, while t deforms the statistics. For $t=-1$ and $t=1$ we obtain structures such as $U_q(g\ell(1|1))$ and $X_q(2)$, all as continuous deformations of $U(g\ell(1|1))$. It is interesting to note that, in the limit of $q=1$, one obtains the braided version of $U(g\ell(1|1))$. Although we have introduced braid statistics in the context of quantum algebras, this example shows that such structures also exist in the classical case. Further examples will be reported elsewhere [27, 33].

6. Conclusions

We have examined the introduction of braided statistics for quantum groups, in the specific context of a generalization of Manin's supersymmetric $E_q(n)$ to P_μ-statistics. Consideration of coactions on quantum spaces makes it clear that the structure arises from compatibility of the quantum spaces, related to a solution of the braid relation, and the braid statistics, which may in general also be taken to arise from a braid solution as in (3.31). This viewpoint suggests obvious generalizations. Our specific example resolves any confusion about the algebras $GL_q(1|1)$ ($U_q(g\ell(1|1))$) and $\hat{X}_q(2)$ ($X_q(2)$) showing that both are points in a two-parameter continuum of deformations of $GL(1|1)$ ($U(g\ell(1|1))$). Majid's approach of superization, which discretely relates these algebras, can be generalized to structures with P_μ-statistics. These results will be reported elsewhere [27, 33]. The possibility of having a braided version of the QISM, in the case of the Perk–Schultz model, needs to be further explored. Finally the question of the quasitriangularity of $U_{q,\mu}(n)$ needs to be addressed.

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